Journal of Approximation Theory **119**, 1–17 (2002)
doi:10.1006/jath.2002.3707

On Strong Approximations of USC Nonconvex-Valued Mappings

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Communicated by Frank Deutsch

Received October 19, 2000; accepted in revised form June 26, 2002

For any upper semicontinuous and compact-valued (usco) mapping $F: X \to Y$ from a metric space X without isolated points into a normed space Y, we prove the existence of a single-valued continuous mapping $f: X \to Y$ such that the Hausdorff distance between graphs Γ_F and Γ_f is arbitrarily small, whenever "measure of nonconvexity" of values of F admits an appropriate common upper estimate. Hence, we prove a version of the Beer–Cellina theorem, under controlled withdrawal of convexity of values of multifunctions. We also give conditions for such strong approximability of star-shaped-valued upper semicontinuous (usc) multifunctions in comparison with Beer's result for Hausdorff continuous star-shaped-valued multifunctions. © 2002 Elsevier Science (USA)

Key Words: multivalued mapping; approximation; selection; function of nonconvexity; paraconvexity; Hausdorff distance.

0. INTRODUCTION

Cellina [3] has proved that an arbitrary upper semicontinuous (usc) convex-valued mapping $F: X \to Y$ from a metric space X into a normed space Y is approximable in the sense that for each $\varepsilon > 0$ the graph Γ_f of some appropriate single-valued continuous mapping $f: X \to Y$ lies in the

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0021-9045/02 $35.00

$(\varepsilon \times \varepsilon)$-neighborhood $\mathcal{O}_\varepsilon(\Gamma_F)$ of the graph Γ_F of F. Subsequently, Cellina [4] further proved that in the case of a convex domain X and a compact-valued mapping F, one can find a *strong* ε-approximation f of F, i.e. one can additionally assume that the symmetric inclusion $\Gamma_F \subset \mathcal{O}_\varepsilon(\Gamma_f)$ holds. Beer [1] showed that the convexity assumption for domain X is in fact not necessary: strong approximation always exists for an arbitrary metric domain with no isolated points or equivalently, for mappings F which map isolated points to singletons.

The following problem arises naturally: Is it possible to omit or replace the convexity assumption for values $F(x)$ of the usc mapping F? Beer [2] has proposed a positive answer for star-shaped-valued mappings $F : X \to Y$. However, such additional freedom for values of F leads to new restrictions for the type of continuity of F. In Beer's theorem [2], F is *continuous* in the Hausdorff sense, and not just a usc mapping.

In the present paper, we prove a Beer–Cellina-type theorem (Theorem 1.1) for the so-called paraconvex-valued mappings F, i.e. for mappings with values $F(x) \subset Y$ whose *functions of nonconvexity* are less than unity. We derive Theorem 1.1 from three rather independent facts.

First, we show (Theorem 1.2) that the solvability of the usual approximability problem with some additional extension-type property always implies the solvability of strong approximation problem, whenever we consider mappings with $UV^{\mathfrak{P}}$-values, where $\mathfrak{P}$ stands for the class of all paracompact spaces. Second, we prove (Theorem 1.3) that bounded paraconvex sets are $UV^{\mathfrak{P}}$-subsets of a normed range space. Third, we extend the results of [13] and show (Theorem 1.4) the necessary extension-approximability property for paraconvex-valued mappings.

Finally, as examples we include the result that Lipschitz transverse perturbation of a convex closed set along an additional direction yields a paraconvex set (Proposition 1.5) and that a certain type of inside perturbation of a closed ball also leads to a paraconvex (and simultaneously, to star-shaped) set (Proposition 1.6). Hence, for the last class of star-shaped-valued mappings we prove Beer's theorem [2] for usc (in general, not continuous) mappings.

1. MAIN RESULTS AND PRELIMINARIES

We denote by $D(m, r)$ the open ball of radius r, centered at the point $m \in M$ in the metric space (M, d), and for a subset $P \subset M$, we denote by $D(P, r)$ the set $\bigcup\{D(p, r) \mid p \in P\}$. The inequality $Hausd(P, Q) < r$ below means that $P \subset D(Q, r)$ and $Q \subset D(P, r)$. The Cartesian product $X \times Y$ of metric spaces will usually be endowed by the max-metric, i.e. $dist((x_1, y_1), (x_2, y_2)) = \max\{dist(x_1, x_2), dist(y_1, y_2)\}$.

For a nonempty subset $P \subset Y$ of a normed space Y, and for an open r-ball $D_r \subset Y$ we define the *relative precision* of an approximation of P by elements of D_r as follows:

$$\delta(P, D_r) = \sup\{dist(q, P)/r | q \in conv(P \cap D_r)\}.$$

For a nonempty subset $P \subset Y$ of a normed space Y the *function* $\alpha_P(\cdot)$ *of nonconvexity* of P associates the following nonnegative number to each positive number r:

$$\alpha_P(r) = \sup\{\delta(P, D_r) \mid D_r \text{ is an open } r\text{-ball}\}.$$

Clearly, the identical equality $\alpha_P(\cdot) \equiv 0$ is equivalent to *convexity* of the closed set P. By Michael [8], the closed set P is said to be *q-paraconvex*, whenever the number q majorates the function $\alpha_P(\cdot)$ and $P \subset Y$ is said to be *paraconvex* if it is q-paraconvex, for some $q < 1$.

Recall that a multivalued mapping $F : X \to Y$ between topological spaces is called *upper semicontinuous* (resp. *lower semicontinuous*, lsc) if for each open $U \subset Y$, its *small* (resp. *full*) preimage, i.e. the set $F_{-1}(U) = \{x \in X \mid F(x) \subset U\}$ (resp. $F^{-1}(U) = \{x \in X \mid F(x) \cap U \neq \emptyset\}$), is open in X. Recall also that a single-valued mapping $f : X \to Y$ is called a *selection* (resp. an *ε-selection*) of a multivalued mapping $F : X \to Y$ if $f(x) \in F(x)$ (resp. $dist(f(x), F(x)) < \varepsilon$), for all $x \in X$. We shall use the abbreviation "usco" for upper semicontinuous compact-valued mappings. We also say that $F : X \to Y$ is a *paraconvex* mapping if all values $F(x), x \in X$, are q-paraconvex sets, for some $q \in [0, 1)$.

Michael [8] proved a selection theorem for paraconvex lsc mappings of paracompact domains (see [9] for a possible substitution of a suitable *functional* majorant instead of the constant q). As a corollary, every paraconvex set is contractible and moreover, it is an absolute retract (AE) with respect to the class $\mathfrak{P}$ of all paracompact spaces. Note that by [12], every metric ε-neighborhood of a paraconvex set, in any uniformly convex space Y, is also a paraconvex set, and hence is an AE.

THEOREM 1.1. *Let $F : X \to Y$ be a usco paraconvex mapping from a metric space X without isolated points into a normed space Y. Then for every $\varepsilon > 0$ there exists a strong ε-approximation of F, i.e. a single-valued continuous mapping $f : X \to Y$ such that $Hausd(\Gamma_F, \Gamma_f) < \varepsilon$.*

We say that a subspace $P \subset Y$ is a *UV^X-subset* of a space Y if for every open $U \supset P$ there exists an open V such that $U \supset V \supset P$ and for every closed subset $A \subset X$, every continuous single-valued mapping $h : A \to V$ admits a continuous single-valued extension $\hat{h} : X \to U$. If open neighborhoods U

and V of a subset $P \subset Y$ are replaced by its open metric ε- and δ-neighborhoods, then we say that $P \subset Y$ is a *metric* UV^X*-subset* of Y.

Clearly, for compacta $P \subset Y$ there is no difference between the notions of metric UV^X-subsets and UV^X-subsets. By considering in these definitions the cases when $\{X = B\}$ are finite-dimensional balls and $\{A\}$ are their boundary spheres, we obtain the standard notion of UV^∞-subsets of Y (see [7]). We say that $P \subset Y$ is a $UV^{\mathfrak{X}}$-subset whenever P is a UV^X-subset of Y, for each X from the class $\mathfrak{X}$ of topological spaces.

For the proof of Theorem 1.1 we use not only metric $(\varepsilon \times \varepsilon)$-approximations, but also topological $(\omega \times \nu)$-approximations. Let ω be an open covering of a topological space X and, respectively, ν an open covering of a topological space Y. We say that $f : X \to Y$ is an $(\omega \times \nu)$-*approximation* of $F : X \to Y$ if for each point $(x, f(x)) \in \Gamma_f$ there exists a point $(x', y') \in \Gamma_F$ such that the points x and x' lie in some element of ω and, respectively, the points $f(x)$ and y' lie in some element of ν. In short,

$$\Gamma_f \subset \mathcal{O}_{\omega \times \nu}(\Gamma_F).$$

The term $(\omega \times \varepsilon)$-*approximation*, where $\varepsilon > 0$, means that we consider the $(\omega \times \nu_\varepsilon)$-approximation with ν_ε the covering of the range metric space Y by the family of all open $(\varepsilon/2)$-balls.

We say that a mapping $F : X \to Y$ is *ES-approximable with respect to a subset* $Z \subset X$ if for all coverings ω of X and ν of Y, every selection $f : Z \to Y$ of the restriction $F|_Z$ admits an extension $\hat{f} : X \to Y$ which is an $(\omega \times \nu)$-approximation of F. Here, the *ES*-abbreviation stands for "extension of selections". For the empty subset Z this notion coincides with the usual approximability of F. The following purely topological theorem reduces (under some additional assumption) the problem of strong approximability to the problem of approximability.

THEOREM 1.2. *Let $F : X \to Y$ be a usco UV^X-valued mapping from a paracompact space X without isolated points into a paracompact space Y and let F be ES-approximable with respect to each discrete closed subset $Z \subset X$. Then F is topologically strongly approximable, i.e. for all coverings ω of X and ν of Y there exists $f : X \to Y$ such that*

$$\Gamma_f \subset \mathcal{O}_{\omega \times \nu}(\Gamma_F) \qquad \textit{and} \qquad \Gamma_F \subset \mathcal{O}_{\omega \times \nu}(\Gamma_f).$$

Theorem 1.2 and the notion of *ES*-approximability have natural metric versions and in fact, for the proof of Theorem 1.1, we really use only such metric facts. The next two theorems show that Theorem 1.2 is indeed applicable for paraconvex mappings.

THEOREM 1.3. *Every bounded paraconvex subset of a normed space is a metric $UV^{\mathfrak{P}}$-subset of the space. In particular, every paraconvex subcompactum of a normed space is a $UV^{\mathfrak{P}}$-subset of the space.*

THEOREM 1.4. *Let $F: X \to Y$ be a usc paraconvex mapping from a metric space X into a normed space Y. Then for every closed discrete subset $Z \subset X$ and every $\varepsilon > 0$, each selection $f: Z \to Y$ of the restriction $F|_Z$ admits an extension $\hat{f}: X \to Y$ which is an ε-approximation of F.*

A typical example of a paraconvex set obtained under some transversal perturbation of a closed convex set is given by the following proposition:

PROPOSITION 1.5. *Let V be a closed convex subset of a uniformly convex space Z, let the Cartesian product $Y = Z \times \mathbb{R}$ be endowed with the norm $||(z,t)|| = \sqrt{||z||_B^2 + t^2}$ and let $\varphi: V \to \mathbb{R}$ be a Lipschitz (with constant L) mapping. Then there exists a number $q = q(L) \in [0,1)$ independent of V and φ, such that the graph Γ_φ of such numerical mapping is a q-paraconvex subset of Y.*

Symmetrically, the following proposition deals with the "inside" perturbation of convex sets. We recall that the *gap* between subsets B and C of a metric space (X,d) is defined as $\inf\{d(b,c) | b \in B, c \in C\}$ and the cone $cone(C; c_0)$ generated by C and centered at c_0 is defined as the set of all sums $c_0 + \sum_{i=1}^n \lambda_i (c_i - c_0)$ over all natural n, all nonnegative λ_i and all $c_i \in C$.

PROPOSITION 1.6. *Let $\{C_\alpha\}_{\alpha \in A}$ be a family of closed convex subsets of a uniformly convex space Y. Let there exist a point $c_0 \in \bigcap_{\alpha \in A} C_\alpha$ and positive numbers τ and s such that all pairwise gaps between $cone(C_{\alpha_1}; c_0) \cap S_\tau$ and $cone(C_{\alpha_2}; c_0) \cap S_\tau$ are greater than or equal to s, $\alpha_1 \neq \alpha_2$, where S_τ stands for the boundary sphere of the closed ball B_τ of radius τ centered at the point c_0. Then the unions $B_\tau \cup (\bigcup_{\alpha \in A} C_\alpha)$ and $\bigcup_{\alpha \in A} C_\alpha$ are paraconvex subsets of Y.*

Note that Proposition 1.6 is false under the replacement of $cone(C_\alpha; c_0)$ by C_α in the separation assumption. In fact, let us consider in the classical Hilbert space l_2 of square-summable sequences with standard basis $\{e_1, e_2, e_3, \ldots\}$ being the family of triangles

$$C_{2n-1} = conv\left\{0; \frac{\sqrt{2}}{2} e_{2n} + \frac{\sqrt{2}}{2} e_{2n-1}; \varepsilon_n e_{2n} + \frac{\sqrt{2}}{2} e_{2n-1}\right\},$$

$$C_{2n} = conv\left\{0; -\frac{\sqrt{2}}{2} e_{2n} + \frac{\sqrt{2}}{2} e_{2n-1}; -\varepsilon_n e_{2n} + \frac{\sqrt{2}}{2} e_{2n-1}\right\}$$

with a sequence of positive $\{\varepsilon_n\}$ tending to zero. Then the star-shaped set $C = \bigcup_{n=1}^{\infty} C_n$ fails to be paraconvex, while the gap condition for the family $\{C_n\}_{n=1}^{\infty}$ with respect to the unit sphere holds.

Finally, as a simple corollary of Lemma 3.2 (see below) we formulate the following, possibly also a new result:

COROLLARY 3.3. *In the Stone–Weierstrass theorem on approximation of continuous functions f by polynomials v (or by elements of some suitable subalgebra of functions) we can always assume that the resulting polynomial v continuously depends on f and on the precision of approximation.*

2. PROOF OF THEOREM 1.2

Let two coverings ω of X and ν of Y be given. For every $x \in X$, the star

$$St(F(x), \nu) = \bigcup \{V \in \nu \mid F(x) \cap V \neq \emptyset\}$$

of the set $F(x)$ with respect to the covering ν constitutes an open neighborhood of the UV^X-subset $F(x) \subset Y$. Hence, there exists an open set $V_x \subset Y$ such that $St(F(x), \nu) \supset V_x \supset F(x)$ and for every closed subset $A \subset X$ every continuous single-valued mapping $h : A \to V_x$ admits a continuous single-valued extension $\hat{h} : X \to St(F(x), \nu)$. So, the family

$$F_{-1}(V_x) = \{x' \in X \mid F(x') \subset V_x\}, \qquad x \in X,$$

of small preimages of the sets $V_x, x \in X$, is the open covering of the paracompact space X.

Clearly, any $(\omega' \times \nu)$-approximation of F is its $(\omega \times \nu)$-approximation, whenever a covering ω' refines the covering ω. So, by virtue of paracompactness of X we can assume that ω consists of the vertical sections of some open neighborhood of the diagonal in $X \times X$:

$$\omega = \{W_x \mid x \in W_x\}_{x \in X}.$$

Using paracompactness of the domain X once more, we fix a locally finite open covering $\mathcal{O} = \{\mathcal{O}_\gamma\}_{\gamma \in \Gamma}$ of X which strongly star-refines the covering $\{F_{-1}(V_x) \cap W_x\}_{x \in X}$, i.e for each index $\gamma \in \Gamma$ one can pick a point $x_\gamma \in X$ such that

$$St(\mathcal{O}_\gamma, \mathcal{O}) \subset F_{-1}(V_{x_\gamma}) \cap W_{x_\gamma}.$$

By the axiom of choice, we can assume that

$$\gamma \neq \gamma' \Rightarrow \mathcal{O}_\gamma \neq \mathcal{O}_{\gamma'}.$$

By Cellina's Lemma 1 from [4], we can pick a family of points $\{z_\gamma\}_{\gamma\in\Gamma}$ such that $z_\gamma \in \mathcal{O}_\gamma,\ \gamma \in \Gamma$ and

$$(\gamma \neq \gamma') \Rightarrow (z_\gamma \neq z_{\gamma'}).$$

The local finiteness of the covering $\mathcal{O}$ guarantees that the set $Z = \{z_\gamma\}_{\gamma\in\Gamma}$ is discrete and closed in X. For each index $\gamma \in \Gamma$ we simply put

$$g(z_\gamma) \in F(z_\gamma) \subset Y.$$

In this way, a continuous selection $g : Z \to Y$ of $F|_Z$ is defined. By assumption, the mapping $F : X \to Y$ is ES-approximable with respect to $Z \subset X$. Therefore, we can continuously extend g over the entire domain X such that $g : X \to Y$ is an $(\mathcal{O} \times v)$-approximation of F. We reserve the notation g for such an extension.

By construction,

$$(z_\gamma \in \mathcal{O}_\gamma \subset F_{-1}(V_{x_\gamma})) \Rightarrow (g(z_\gamma) \in F(z_\gamma) \subset V_{x_\gamma}).$$

By continuity of g we can choose for each index $\gamma \in G$, a neighborhood N_γ of the point z_γ such that $g([N_\gamma]) \subset V_{x_\gamma}$ and $[N_\gamma] \subset \mathcal{O}_\gamma$, where $[N_\gamma]$ is the closure of N_γ. By passing to subneighborhoods, we can assume that the family $\{[N_\gamma]\}_{\gamma\in\Gamma}$ is disjoint due to the discreteness of the set Z. Moreover, the domain X has no isolated points and this is why we can consider neighborhoods with nonempty boundary sets $B_\gamma = [N_\gamma] \backslash N_\gamma,\ \gamma \in \Gamma$. We preserve the $(\mathcal{O} \times v)$-approximation $g : X \to Y$ outside the disjoint union $\bigsqcup_{\gamma\in\Gamma} [N_\gamma]$ and change it inside each open set $N_\gamma,\ \gamma \in \Gamma$.

To this end, following the idea of Beer [1], we choose an appropriate finite net in the compact set $F(x_\gamma) \subset Y$. More precisely, we consider all nonempty intersections of the compactum $F(x_\gamma)$ with elements of the covering v. Find a locally finite open strong star-refinement, say λ_γ of this covering, fix a finite subcovering of the covering λ_γ and pick a single point at each element of this finite subcovering. So, we construct a finite subset

$$Y_\gamma = \{y_{\gamma,1}, y_{\gamma,2}, \ldots, y_{\gamma,n_\gamma}\} \subset F(x_\gamma)$$

such that for every $y \in F(x_\gamma)$, there exists $y_{\gamma,i} \in Y_\gamma$ which lies together with y in some element of the covering v. That is, y and $y_{\gamma,i}$ are v-close. No point

$z_\gamma \in N_\gamma \subset X$ is isolated. Hence, there exists a finite set

$$Z_\gamma = \{z_{\gamma,1}, z_{\gamma,2}, \ldots, z_{\gamma,n_\gamma}\} \subset N_\gamma$$

of pairwise different points.

We now put $A_\gamma = B_\gamma \sqcup Z_\gamma,\ \gamma \in \Gamma$ and

$$h_\gamma(a) = \begin{cases} g(a), & a \in B_\gamma, \\ y_{\gamma,i}, & a = z_{\gamma,i} \in Z_\gamma. \end{cases}$$

Clearly, $h_\gamma : A_\gamma \to Y$ is a continuous mapping defined on the closed subset A_γ of the paracompact space $[N_\gamma]$. Moreover, by construction, all values of h_γ lie in the open neighborhood V_{x_γ} of the UV^X-subset $F(x_\gamma) \subset Y$. Therefore, there exists an extension $\hat{h}_\gamma$ of h_γ which continuously maps $[N_\gamma]$ into the star $St(F(x_\gamma), v)$ of the set $F(x_\gamma)$ with respect to the covering v. Finally, we define the continuous mapping $f : X \to Y$ by setting

$$f(x) = \begin{cases} g(x), & x \notin \bigsqcup_{\gamma\in\Gamma}[N_\gamma], \\ \hat{h}_\gamma(x), & x \in [N_\gamma]. \end{cases}$$

The mapping f coincides with g over $X \setminus \bigsqcup[N_\gamma]$ and hence the graph of its restriction lies in the $(\mathcal{O} \times v)$-neighborhood (and, hence in the $(\omega \times v)$-neighborhood) of the graph Γ_F. For each $\gamma \in \Gamma$, the set $[N_\gamma] \subset \mathcal{O}_\gamma$ is the subset of the element W_{x_γ} of the covering ω and $x_\gamma \in W_{x_\gamma}$. In the range space Y, we see that $f([N_\gamma]) = \hat{h}_\gamma([N_\gamma]) \subset St(F(x_\gamma), v)$. Hence, the graph of f over each set $[N_\gamma]$ also lies in the $(\omega \times v)$-neighborhood of the graph Γ_F.

For the checking of symmetric inclusion $\Gamma_F \subset \mathcal{O}_{\omega\times=v}(\Gamma_f)$ let us pick $(x, y) \in \Gamma_F$. For some index $\gamma \in \Gamma$ we see that $x \in \mathcal{O}_\gamma \subset F_{-1}(V_{x_\gamma}) \cap W_{x_\gamma}$. So, the point x is ω-close to each of the points $\{z_{\gamma,1}, z_{\gamma,2}, \ldots, z_{\gamma,n_\gamma}\}$. In the range space Y we see that $y \in F(x) \subset V_{x_\gamma} \subset St(F(x_\gamma), v)$. So y is v-close to some point, say y', of the compactum $F(x_\gamma) \subset Y$. But y' is v-close to some point $y_{\gamma,i} \in F(x_\gamma)$ chosen above. Hence, y is $St(v)$-close to the chosen point $y_{\gamma,i} = f(z_{\gamma,i})$. To complete the proof it suffices to perform the entire constructions above not exactly for the given covering v of Y, but for its arbitrary star-refinement v'. Then $\{y, y_{\gamma,i}\} \subset St(y', v') \subset V$, for some $V \in v$. Theorem 1.2 is thus proved. ∎

3. PROOFS OF THEOREMS 1.3 AND 1.4

Small perturbations in the sense of Hausdorff distance of a paraconvex set P unfortunately yield nonparaconvex sets. To avoid such instability, we introduce the following notion. For $\varepsilon \geqslant 0$ and for $q \in [0, 1)$, a subset P of a normed space Y is said to be *q-paraconvex with precision* ε if $\alpha_P(r) \leqslant q$, for all $r > \varepsilon$.

First, we state the following stability property for paraconvexity with prescribed positive precision. For the sake of completeness we reproduce the proof from [13].

PROPOSITION 3.1. *For every normed space Y, every $q \in [0,1), \varepsilon > 0$, $p \in (q,1)$ there exists $\lambda \in (0,\varepsilon)$ such that for every q-paraconvex with precision ε subset $P \subset Y$ and for every $Q \subset Y$ with $Hausd(P,Q) < \lambda$, the subset Q is p-paraconvex with precision ε.*

Proof. We show that one can put $\lambda = \varepsilon\frac{p-q}{6}$. Let $r > \varepsilon$ and let an open ball D_r intersect Q. For $y \in conv\{y_1,\dots,y_n\}$, $y_i \in D_r \cap Q$, one can choose $z_i \in D_{r+\lambda} \cap P$ with $dist(y_i,z_i) < \lambda$, where $D_{r+\lambda}$ is the ball concentric with D_r. Due to the convexity of λ-balls there exists $z \in conv\{z_1,\dots,z_n\}$ with $dist(z,y) < \lambda$. So,

$$dist(z,P) \leqslant \alpha_P(r+\lambda)\cdot(r+\lambda) \leqslant qt(r+\lambda) < q'(r+\lambda), \qquad q' = \frac{p+q}{2}.$$

Pick $z_0 \in P$ with $dist(z,z_0) < q'(r+\lambda)$ and find $y_0 \in Q$ with $dist(z_0,y_0) < \lambda$. Then

$$dist(y,Q) \leqslant dist(y,y_0) \leqslant dist(y,z) + dist(z,z_0) + dist(z_0,y_0) < 2\lambda + q'(r+\lambda).$$

To complete the proof it suffices to verify that $2\lambda + q'(r+\lambda) < pr$ or $\lambda(2+q') < (p-q')r$ or

$$\lambda < r\frac{p-q'}{2+q'} = r\frac{p-q}{4+p+q}.$$

Finally, we see that

$$\lambda = \varepsilon\frac{p-q}{6} < \varepsilon\frac{p-q}{4+p+q} < r\frac{p-q}{4+p+q}. \quad \blacksquare$$

In the proof of Theorem 1.3, we use the Michael selection theorem for convex-valued mappings into Banach spaces [11]. So, we must be more careful with relations between normed spaces and Banach spaces.

LEMMA 3.2. *Let B be the completion of the normed space Y. Then there exists a continuous mapping $b : B \times (0,\infty) \to Y$ such that*

$$||y - b(y,r)|| < r$$

for all $(y,r) \in B \times (0,\infty)$.

As an example, we have Corollary 3.3 (see Section 1) concerning the Stone–Weierstrass theorem.

Proof of Lemma 3.2. For a fixed $r > 0$, we consider the covering ω of the whole Banach space B by the open balls $D(y', r)$ of radius r, centered at the points $y' \in Y$. Let $\{e_\alpha\}$ be a locally finite continuous partition of unity inscribed into the covering ω. For any index α, we pick an element $y_\alpha \in Y$ such that support $supp\, e_\alpha$ of the continuous function $e_\alpha : B \to [0, 1]$ is a subset of the ball $D(y_\alpha, r)$. In the standard manner we put

$$b(y, r) = \sum_\alpha e_\alpha(y) y_\alpha \in Y, \quad y \in B.$$

By the local finiteness of $\{e_\alpha\}$ and convexity of balls, we see that for some finite number of indices

$$||y - b(y, r)|| = \left|\left|\sum_{i=1}^{n} e_{\alpha(i)}(y)(y - y_{\alpha(i)})\right|\right| \leqslant \sum_{i=1}^{n} e_{\alpha(i)}(y)||(y - y_{\alpha(i)})|| < r.$$

One can make such a procedure for a sequence $r_n = 1/n$, $n \in \mathbb{N}$, and then draw the straight line segments $[b(y, r_n), b(y, r_{n+1})]$. To complete the proof it suffices for $r = (1 - t)r_{n+1} + tr_n$, $t \in [0, 1)$, to set

$$b(y, r) = (1 - t)b(y, r_{n+2}) + tb(y, r_{n+1}).$$

Lemma 3.2 is thus proved. ∎

Proof of Theorem 1.3. Let P be a bounded q-paraconvex subset of a normed space Y and let $\varepsilon > 0$ be a given radius of the metric neighborhood $D(P, \varepsilon)$ of P. Clearly, P is q-paraconvex with precision $\varepsilon/2$. Pick $p \in (q, 1)$. By Proposition 3.1, one can find a positive number $\lambda = \lambda(q, \varepsilon/2, p)$ such that the inequality $Hausd(P, Q) < \lambda$ implies the p-paraconvexity of the set Q with precision $\varepsilon/2$. In particular, the closed $(\lambda/2)$-neighborhood $Q = Cl(D(P, \lambda/2))$ is p-paraconvex with precision $\varepsilon/2$ subset of Y.

Now for every paracompact space X, its closed subset $A \subset X$ and every continuous single-valued mapping $h : A \to V = D(P, \lambda/2)$ we want to find a continuous single-valued extension $\hat{h} : X \to U = D(P, \varepsilon)$. By Lemma 3.2, it suffices to extend the mapping $h : A \to V = D(P, \lambda/2)$ to some mapping $h' : X \to U' = D_B(P, 3\varepsilon/4)$ and then set $\hat{h} = b' \circ h'$, where $b' = b(\cdot, \varepsilon/4)$, $b' : B \to Y$.

Let us consider closed convex bounded subset $C = Cl_B(conv\, Q)$ of the Banach space B. The continuous mapping $h : A \to V = D_Y(P, \lambda/2) =\subset Q \subset C$ admits a continuous extension, say $h_0 : X \to C$, by the classical Michael selection theorem. All values of such extensions are R-close to the set Q, for some sufficiently large $R > 0$.

If $R \leqslant \varepsilon/2$ then we can simply put $h' = h_0$. Otherwise, one can use p-paraconvexity with precision $\varepsilon/2$ of the set Q. Namely, if

$$H_1(x) = \begin{cases} Cl_B(conv)\{Q \cap D(h_0(x), R)\}, & x \notin A, \\ \{h(x)\}, & x \in A. \end{cases}$$

Then the Michael selection theorem can be applied to mapping H_1. Hence, for a selection h_1 of H_1 we have

$$dist(h_1(x), Q) \leqslant \alpha_Q(R)R < p'R = R_1$$

for some $p' \in (p, 1)$. So, all values of $h_1 : X \to B$ are R_1-close to the set Q.

If $R_1 \leqslant \varepsilon/2$ then we can put $h' = h_1$. Otherwise we repeat the above construction and find a continuous extension $h_2 : X \to B$ of h such that all values of h_2 are R_2-close to the set Q; $R_2 = (p')^2 R$. After some finite number N of similar steps we obtain a continuous extension $h' = h_N : X \to B$ of h such that all values of h' are $\varepsilon/2$-close to the set Q. But $Hausd(P, Q) < \lambda < \varepsilon/4$, see exact answer for λ from Proposition 3.1. Thus all values of h' are $(3\varepsilon/4)$-close to the set P, i.e. the mapping h' really maps paracompact domain X into the neighborhood $U' = D_B(P, 3\varepsilon/4)$. Theorem 1.3 is thus proved. ∎

Proof of Theorem 1.4. Here we in fact generalize Theorem 6 from [13], the proof of which shows that for every paraconvex-valued usc mapping F from a metric space X into a normed space Y, for each covering ω of X and each $\varepsilon > 0$, there exists an $(\omega \times \varepsilon)$-approximation, say g, of F.

We repeat the UV-technique from the proof of Theorem 1.2 for making some surgery of g in order to obtain an $(\varepsilon \times \varepsilon)$-approximation $\hat{f} : X \to Y$ with chosen values $\hat{f}(z) = f(z) \in F(z)$, for all z from the given discrete closed subset $Z \subset X$. In comparison with Theorem 1.2, we do not construct an appropriate Z, but in the converse direction, we work with the preassigned $Z \subset X$.

For each $x \in X$ and each ε-neighborhood $D(F(x), \varepsilon)$, pick an open neighborhood V_x of the set $F(x)$ with respect to the UV-property of paraconvex set $F(x)$, see Theorem 1.3. Moreover, the proof of Theorem 1.3 shows that it is possible to put $V_x = D(F(x), 2\delta)$ with $\delta = \varepsilon(1 - q)/50$ independent of the variable x.

As in the proof of Theorem 1.2, fix a strong star-refinement $\mathcal{O}$ of the covering

$$\{F_{-1}(D(F(x), \delta)) \cap D(x, \varepsilon/2)\}_{x \in X}.$$

Clearly, we can additionally assume the disjointness of the stars of elements of the given discrete closed set Z with respect to covering $\mathcal{O}$. So, let $g : X \to Y$ be an $(\mathcal{O} \times \delta)$-approximation of F. We preserve g outside the union of disjoint stars $St(z, \mathcal{O}), z \in Z$, and change it inside these stars. Therefore, we in fact consider the case of the single point $z \in Z$. If z is an isolated point of X then we simply put $\hat{f}(z) = f(z)$.

Let z be a nonisolated point. There exists a point $\hat{x} \in X$ such that $\hat{x}$ is $\mathcal{O}$-close to z and $g(z)$ is δ-close to the set $F(\hat{x})$. So, there exists another point, say $x \in X$, such that

$$St(z, \mathcal{O}) \subset F_{-1}(D(F(x), \delta)) \cap D(x, \varepsilon/2).$$

Therefore,

$$F(\hat{x}) \subset F(St(z, \mathcal{O})) \subset D(F(x), \delta)$$

and

$$g(z) \in D(F(x), 2\delta).$$

Hence, $g(N_z) \subset D(F(x), 2\delta)$ for some neighborhood $N_z \subset St(z, \mathcal{O})$ of the point z. Of course, we see that $f(z) \in F(z) \subset D(F(x), 2\delta)$, too.

Clearly, we can assume that the boundary of N_z is nonempty. So, let A be the union of the boundary of N_z and the point z. Then A is a closed subset of $Cl(N_z)$. Consider the mapping, say $h : A \to Y$, which coincides with g over the boundary of N_z and which associates to z the point $f(z) \in F(z)$. By the UV-property of the set $F(x)$ and choice of δ, we can extend h to some mapping $\hat{h} : Cl(N_z) \to D(F(x), \varepsilon)$. Clearly, the graph $\Gamma_{\hat{h}}$ of such an extension is $(\varepsilon \times \varepsilon)$-close to the graph Γ_F. By performing such a surgery at each point $z \in Z$, we obtain the desired extension $\hat{f}$ of f. ∎

4. PROOFS OF PROPOSITIONS 1.5 AND 1.6

For simplicity we consider only the case of inner product spaces Y. For one-dimensional space Y, Proposition 1.5 was proved in [9] and for finite-dimensional spaces in [10]. Here, we generalize these results to an arbitrary Y and, on the other hand, give a new and simpler approach in comparison with the technically complicated proof from [10].

Proof of Proposition 1.5. Let $z_1, z_2, \ldots, z_n$ be any points of V and $y_i = (z_i, \varphi(z_i))$ be the corresponding points on the graph Γ_φ. We estimate the distance $dist(c, \Gamma_\varphi)$ for the (unique) Chebysheff center c of the polygon

$P = conv\{y_1, y_2, \ldots, y_n\}$ via the Lipschitz constant L and the Chebysheff radius $R = R(P)$ of this polygon.

If c lies on the boundary of P then we can pass to a smaller dimension and argue inductively. If c is an inner point of P then all distances $d(c, y_i)$ are equal to R.

Draw via the point c the "horizontal" hyperplane Π (i.e. Π is parallel to Z). Denote by c^* the point of the graph Γ_φ, corresponding to c. If $c \in \Pi$, then $c = c^* \in \Gamma_\varphi$ and hence $dist(c, \Gamma_\varphi) = 0$. If $c \notin \Pi$, then the hyperplane Π separates the point c^* and one of the points $y_1, y_2, \ldots, y_n$; say the point y_1.

Let Π_0 be the two-dimensional plane which passes through the points c, c^*, y_1. Consider the restriction of the function φ on the intersection $V \cap \Pi_0$. In the plane Π_0 draw the angle with the origin at the point y_1, with the horizontal bissectrix intersecting the vertical line cc^* and with measure equal to $2\arctan(L)$. By the choice of the point y_1 and due to the Lipschitz property with the constant L of the function φ, we see that the points c and c^* are in the same half of this angle. So, the graph Γ_φ intersects the perpendicular drawn from the point c to the segment $[y_1, c^*]$. Note that the length of such a perpendicular is less than or equal to $d(c, y_1) \cdot \sin(\arctan(L)) = R \cdot \sin(\arctan(L))$. Hence $dist(c, \Gamma_\varphi) \leqslant R \cdot \sin(\arctan(L))$.

For an inner point $y \in P$ which differs from the Chebysheff center c, the following two cases are possible:

(a) y is close to c and then the estimate $dist(y, \Gamma_\varphi)$ is approximately the same as for $dist(c, \Gamma_\varphi)$;

(b) the distance $d(c, y)$ is greater than some constant and then one of the distances $d(y, y_i)$ will be essentially less than R.

More precisely, if in case (a) the distance $d(c, y)$ is less than or equal to εR, then $dist(y, \Gamma_\varphi) \leqslant d(c, y) + dist(c, \Gamma_\varphi) < (p + \varepsilon)R$, where $p = sin(arctan(L))$. In the second case (b) we have $d(c, y) > \varepsilon R$. Convexity of P implies that for some point y_i the triangle Δcyy_i has an obtuse angle at the vertex y. So,

$$d(y, \Gamma_\varphi)^2 \leqslant d(y, y_i)^2 \leqslant d(c, y_i)^2 - d(c, y)^2 < (1 - \varepsilon^2)R^2.$$

Combining (a) and (b), we obtain that

$$dist(y, \Gamma_\varphi) \leqslant \max\{p + \varepsilon, \sqrt{1 - \varepsilon^2}\}R.$$

So, we can define the parameter $\varepsilon > 0$ as the root of the equation

$$(p + x)^2 = 1 - x^2$$

and put $q = p + \varepsilon \in [0, 1)$. Then each point of the polygon P is qR-close to the graph of the function φ. To complete the proof it now suffices to observe that the Chebysheff radius R of the intersection $D_r \cap \Gamma_\varphi$ is less than or equal to the radius r of the ball D_r. So the function of nonconvexity of the graph Γ_φ majorizes by the constant $q \in [0, 1)$. ∎

We now pass to the "hedgehog"-shaped sets.

Proof of Proposition 1.6. First, we examine the union $C = \bigcup_{\alpha \in A} C_\alpha$. The gap assumption for intersections $cone(C_\alpha; c_0) \cap S_\tau$ with the convexity of all C_α together show that all pairwise intersections $C_{\alpha_1} \cap C_{\alpha_2}$ are equal to $\{c_0\}$.

So let P be a convex hull of a finite subset $\{y_1, y_2, \ldots, y_n\}$ of the star-shaped set C and y a point of the polygon P. We want to estimate the distance $dist(y, C)$. First, we consider the special case where the points y_i belong to distinct C_α. Then we reduce the general case to this special situation.

(1) Let $y_1 \in C_1 = C_{\alpha_1},\ y_2 \in C_2 = C_{\alpha_2}, \ldots, y_n \in C_n = C_{\alpha_n}$ for pairwise different indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $c_0 \notin \{y_1, y_2, \ldots, y_n\}$. Denote by R the Chebysheff radius of P and let c be the Chebysheff center of P. As in the proof of Proposition 1.5, it suffices to check that $dist(c, C) \leqslant pR$, for some $p \in [0, 1)$. Moreover, we show that p can be chosen as a function of parameters τ and s, independent of n and the choice of convex sets C_α.

Thus, let Π be the cone over P with vertex c_0. By the separation hypothesis and by the convexity of the sets C_α, we see that all plane angles $\angle y_i c_0 y_j$ of the "pyramid" Π at the vertex c_0 are greater than or equal to some positive constant β. Moreover, β depends only on the ratio τ/s.

LEMMA 4.1. *Let $y \in P$. Then one of the angles $\angle y y_i c_0$ is less than or equal to* $90^\circ - \frac{\beta}{2}$.

Proof of Lemma 4.1. Let $[c_0, y_1]$ be an edge of maximal length among all edges $[c_0, y_i]$. Consider the flat triangle $\Delta y_1 c_0 y_2$. By the maximality we see that

$$\angle y_2 y_1 c_0 \leqslant \angle y_1 y_2 c_0.$$

But the sum of these two angles is less than or equal to $180^\circ - \beta$. Hence $\beta_2 = \angle y_2 y_1 c_0 \leqslant 90^\circ - \frac{\beta}{2}$. Analogously, we have that $\beta_k = \angle y_k y_1 c_0 \leqslant 90^\circ - \frac{\beta}{2}$, $k = 3, 4, \ldots, n$. But the angle $\angle y y_1 c_0$ is a convex combination of $\beta_2, \beta_3, \ldots, \beta_n$. Therefore, $\angle y y_1 c_0 \leqslant 90^\circ - \frac{\beta}{2}$. Lemma 4.1 is thus proved. ∎

Applying Lemma 4.1 to the case $y = c$ we see that the distance between the Chebysheff center c of P and one of the edges $[c_0, y_i]$ is less than or equal to $d(c, y_i)\sin(90^\circ - \frac{\beta}{2}) = R\cos\frac{\beta}{2}$. But each edge $[c_0, y_i]$ is a subset of the star-shaped set C. Hence, $dist(c, C) \leqslant pR,\ p = \cos\frac{\beta}{2}$. As in the proof of Proposition 1.5 we conclude that for each point $y \in P$, the inequality $dist(y, C) \leqslant qR$ holds for some $q \in [0, 1)$, depending only on p.

(2) In the general situation, let $y_1, y_2, \ldots, y_n$ be arbitrary points of the intersection $D_r \cap C$ of the set C with a ball D_r of radius r and $y = \sum_{i=1}^{n} \lambda_i y_i \in conv\{y_1, y_2, \ldots, y_n\} = P;\ \lambda_i > 0,\ \sum \lambda_i = 1$.

Denote by $C_1 = C_{\alpha_1},\ C_2 = C_{\alpha_2}, \ldots, C_m = C_{\alpha_m};\ 1 \leqslant m \leqslant n$ the convex sets from the given family such that

$$\{y_1, y_2, \ldots, y_n\} \subset C_1 \cup C_2 \cup \cdots \cup C_m.$$

Clearly, one can choose a renumeration $\{1, 2, \ldots, k_1, \ldots, k_2, \ldots, k_{m-1}, \ldots, k_m = n\}$ of indices such that $\{y_1, y_2, \ldots, y_{k_1}\}$ are all elements lying in C_1, $\{y_1, y_2, \ldots, y_{k_2}\}$ are all elements lying in $C_1 \cup C_2, \ldots,$ and $\{y_1, y_2, \ldots, y_{k_{m-1}}\}$ are all elements lying in $C_1 \cup C_2 \cup \cdots \cup C_{m-1}$. Invoking the convexity of the ball D_r and the set C_1 we represent the item $\sum_{i=1}^{k_1} \lambda_i y_i$ as $\mu_1 y_1'$, where

$$\mu_1 = \sum_{i=1}^{k_1} \lambda_i > 0, y_1' = \sum_{i=1}^{k_1} \frac{\lambda_i}{\mu_1} y_i \in conv\{y_1, y_2, \ldots, y_{k_1}\} \subset D_r \cap C_1.$$

By performing such a representation for all $1 \leqslant j \leqslant m$, we see that the point y appears as a convex combination $y = \sum_{j=1}^{m} \mu_j y_j$ of points $y_j \in D_r \cap C_j,\ 1 \leqslant j \leqslant m$. So, we obtain the situation from case (1) and hence $dist(y, C) \leqslant qr$, for some suitable $q \in [0, 1)$. Therefore, the constant q majorizes the function $\alpha_C(\cdot)$ of nonconvexity of the set C. Proposition is thus proved for the union $\bigcup_{\alpha \in A} C_\alpha$.

In the case of $B_\tau \cup (\bigcup_{\alpha \in A} C_\alpha)$, it suffices by the proof of (2), to consider the convex hull $P = conv\{y_1, y_2, \ldots, y_n, y_*\}$ with $y_i \in C_i = C_{\alpha_i}$ and $y_* \in B_\tau \setminus \bigcup C_i$. Defining the conical (with respect to the point c_0) $(\beta/3)$-enlargements C_i' of sets C_i, we see exactly two possibilities for the point y_*.

If $y_* \in \bigcup C_i'$ then case (2) above works with the separation constant $\beta/3$. Otherwise, all angles $\angle y_i c_0 y_*$ are greater than or equal to $\beta/3$. So case (1) above really applies. Proposition 1.6 is thus proved. ∎

5. EPILOGUE

Theorem 1.2 is a purely topological fact, whereas Theorems 1.3 and 1.4, as well as the resulting Theorem 1.1 deal with normed geometry and use the analytical convex techniques. Any attempt to find topological versions of latter theorems is unsuccessful if one considers an arbitrary (even compact) domain of multifunctions. In fact, no approximately invertible cell-like surjection between compacta can increase the Lebesgue dimension [6].

However, the well-known Dranishnikov example [5] shows that there exists a cell-like surjection f of a finite-dimensional compactum onto an infinite-dimensional compactum. Hence $F = f^{-1}$ gives an example of a nonapproximable usc mapping with maximally nice (from topological point of view) values. This is the reason, why we have been working in metric rather than topological terms. In spite of this, we hope that the answer to the following question is affirmative:

Question 5.1. Do there exist purely topological conditions on the domain, the range of a multifunction, and on the family of its values, under which strong approximability is equivalent to approximability?

A substitution of a constant $q \in [0, 1)$ by some suitable function as a majorant for family of functions of nonconvexity works in selection theory (see [9,12]). However, the possibility of such a substitution is unclear for the theory of approximations. For a technical obstruction see the proof of Proposition 3.1: if q depends on the variable x then we cannot find a common positive minorant for the variable λ.

Question 5.2. Is it true that the statement of Theorem 1.1 (or Theorem 1.4) holds for mappings with $\alpha(\cdot)$-paraconvex values, where $\alpha : (0, \infty) \to [0, 1)$ is an increasing function?

Note that Beer [2] gave an example of a continuous continua-valued mapping which is not strongly approximable. All values of this mapping are AE-sets (subarcs of a circle), with a single exception: one of the values is this circle.

As for a purely geometrical question we ask:

Question 5.3. Let $x : [a, b] \to \mathbb{R}$ and $y : [a, b] \to \mathbb{R}$ be two functions with Lipschitz constant L. Is it true that the function of nonconvexity of the curve $\{t, x(t), y(t)\}_{t \in [a,b]}$ considered as the subset of the three-dimensional Euclidean space has a majorant $q \in [0, 1)$ which depends only on L?

ACKNOWLEDGMENTS

Repovš was supported in part by the Ministry of Education, Science and Sport of the Republic of Slovenia Research Program No. 0101-509. Semenov was supported in part by the RFFI Grants No. 99-01-00235 and No. 02-01-00014. We thank the referee for many comments and remarks.

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